

MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
 - Answer **all** questions in Section A and **two** questions from Section B.
 - Calculators are permitted, provided they are silent, self-powered, without communication facilities, and incapable of holding text or other material that could be used to give a candidate an unfair advantage. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
 - Statistical tables are not required.
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Section A

1. Simplify the following expressions:

(a) $\frac{1}{1 + \sqrt{3}} \left(\frac{1}{1 - \frac{1}{\sqrt{3}}} - 1 \right)$, [3 marks]

(b) $\log_6 9 + \log_6 4$, [2 marks]

(c) $\frac{x^2 - x - 2}{3x - 6} + \frac{6 - x^2 - x}{3x + 9}$. [5 marks]

2. Differentiate the following expressions with respect to x , simplifying your answer as far as possible:

(a) $x^5 + \cos x$, [3 marks]

(b) $\frac{1}{2} \ln(1 + x^2)$, [4 marks]

(c) $x(1 + \tan^2(5x)) \cos^2(5x)$. [3 marks]

3. Integrate the following expressions with respect to x :

(a) $e^{3x} + 5x^9$, [3 marks]

(b) $(3x + 1)^2 + \cos x$, [4 marks]

(c) $\frac{9}{3x + 1}$. [3 marks]

4. (a) Write down the equation of the circle C , with centre at $(2, 1)$ and which intersects the x -axis at the point $P_1 = (2, 0)$. [3 marks]

(b) Write down the equation of the line L which intersects C at the point P_1 , and has gradient -2 . [2 marks]

(c) Find the second point P_2 at which the line L intersects C . [5 marks]

(d) Sketch the graphs of C and L on the xy -plane, indicating also P_1 and P_2 . [5 marks]

5. (a) Find all solutions x (in radians) of the equation

$$(2 \tan x \sin x - 1) \cos x = 1$$

that also satisfy $0 < x < 2\pi$. [8 marks]

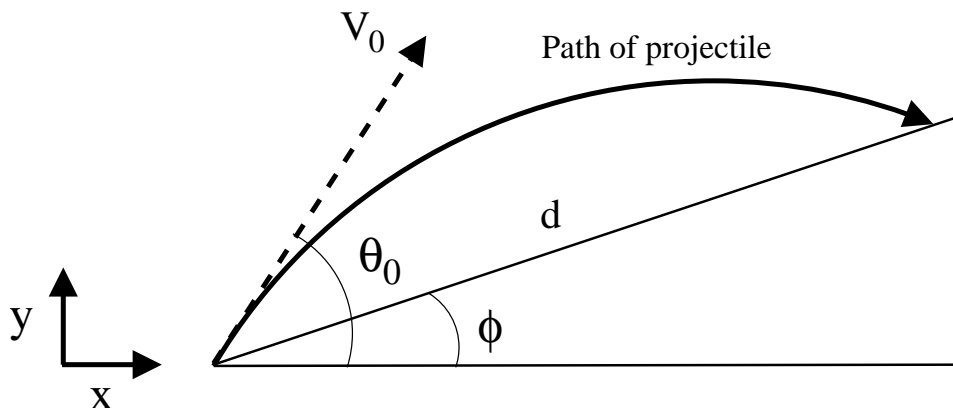
(b) Show that the equation $e^{-3x} - \sin(x^3) = 0$ has at least three roots in the interval $(0, 2)$. (**Note:** you are not required to find the roots.) [7 marks]

Section B

6. Consider the function $f(x) = 1/(1 + x^2)$.

- (a) Find the derivatives $f'(x)$ and $f''(x)$, and find a point $x_0 < 0$ such that $f(x_0) = f'(x_0) = f''(x_0)$. [7 marks]
- (b) Show that the curves $y = f(x)$ and $y = f'(x)$ have the same tangent line L at x_0 , and find the equation of L . [4 marks]
- (c) Show that $f(x) \geq f'(x)$ for every real number x . [3 marks]
- (d) Calculate the area enclosed by the curves $y = f(x)$ and $y = f'(x)$, and the lines $x = x_0$ and $x = 0$. [6 marks]

7. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_0 at angle θ_0 with respect to the horizontal ($\theta_0 > \phi$) as shown in the figure below.



- (a) Show that the trajectory of the projectile is given by the equation

$$y = (\tan \theta_0)x - \frac{g}{2v_0 \cos^2 \theta_0} x^2$$

where g is the gravitational acceleration.

[3 marks]

- (b) Hence show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_0^2 \cos \theta_0 \sin(\theta_0 - \phi)}{g \cos^2 \phi}.$$

[7 marks]

- (c) For what value of θ_0 is d at its maximum, and what is the maximum value? Simplify your answer for the maximum value as far as possible. [10 marks]

You may require the following identities:

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta), \\ \cos(\alpha + \beta) &= \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta), \\ \sin(\alpha) \cos(\beta) &= \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta). \end{aligned}$$

8. Suppose that a continuous random variable, X , has a probability density function given by

$$f(x) = \begin{cases} kx & \text{for } 0 \leq x < 1, \\ 2k - kx & \text{for } 1 \leq x \leq 2, \\ 0 & \text{for } x < 0 \text{ or } x > 2. \end{cases}$$

- (a) Find the value of k . [3 marks]
- (b) Draw a graph of the function $f(x)$ for $-2 \leq x \leq 4$. [2 marks]
- (c) Find $\mathbb{E}(X)$ and $\text{Var}(X)$. [10 marks]
- (d) If $Y = 3X + 2$, find $\mathbb{E}(Y)$ and $\text{Var}(Y)$. [2 marks]
- (e) Find M such that $P(X \leq M) = 0.4$. [3 marks]